Jonas Kremer WS 2017/2018

Probability theory

Exercise Sheet 5

Exercise 1 *(5 Points)*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Let $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$. Prove the following properties of the conditional expectation:

(a) Let $Y \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $a,\ b \in \mathbb{R}$. Then

$$\mathbb{E}\left(aX + bY \mid \mathcal{G}\right) = a\mathbb{E}\left(X \mid \mathcal{G}\right) + b\mathbb{E}\left(Y \mid \mathcal{G}\right).$$

(b) Let $\mathcal{A} \subset \mathcal{G}$ be a further sub-σ-algebras. Then

$$\mathbb{E}\left(\mathbb{E}\left(X \mid \mathcal{G}\right) \mid \mathcal{A}\right) = \mathbb{E}\left(\mathbb{E}\left(X \mid \mathcal{A}\right) \mid \mathcal{G}\right) = \mathbb{E}\left(X \mid \mathcal{A}\right).$$

(c) We have that

$$\mathbb{E}\left(\mathbb{E}\left(X \mid \mathcal{G}\right)\right) = \mathbb{E}(X).$$

(d) Let $Y \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ with $Y \leq X$. Then

$$\mathbb{E}\left(Y \mid \mathcal{G}\right) \leq \mathbb{E}\left(X \mid \mathcal{G}\right).$$

(e) We have that

$$\left|\mathbb{E}\left(X \mid \mathcal{G}\right)\right| \leq \mathbb{E}\left(|X| \mid \mathcal{G}\right).$$

Exercise 2 *(4 Points)*

(a) Let $X,\ Y$ be two random variables with $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$ almost surely. Are X and Y necessarily independent?

(b) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra. Suppose that X and Y are two random variables such that

$$\mathbb{E}\left(Y \mid \mathcal{G}\right) = X \quad \text{and} \quad \mathbb{E}\left(Y^2\right) = \mathbb{E}\left(X^2\right) < \infty.$$

Prove that $X = Y$ almost surely.

Exercise 3 *(4 Points)*
Let X, Y be two random variables with joint density function f, and let $h : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}(|h(X)|) < \infty$. We denote by $m(\mathrm{d}x)$ the Lebesgue measure on $\mathbb{R}$. Show that almost surely:[1]

$$\mathbb{E}\left(h(X) \,|\, Y\right) = \frac{\int h(x) f(x, Y) m(\mathrm{d}x)}{\int f(x, Y) m(\mathrm{d}x)}.$$

Exercise 4 *(4 Points, talk)*
Prepare a talk on the proof of the following statement:
Let (E, d) be a metric space and μ a probability measure on E. Then, for all $A \in \mathcal{B}(E)$,

$$\mu(A) = \sup \left\{ \mu(K) \,|\, K \subset A \text{ closed} \right\} = \inf \left\{ \mu(U) \,|\, A \subset U \text{ open} \right\}.$$

[1] By convention: $0/0 := 0$.